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Generalized Disconjugacy and Comparison Theorems

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We consider the pair of equations $L_n y + py = 0$, $L_\nu y + qy = 0$, where L_n, L_ν are disconjugate differential operators of order n and ν, respectively, and establish some comparison theorems between them.

1. INTRODUCTION

Suppose that the real functions $f_i \in C^{(n-i)}(I)$, $I \subset \mathbb{R}$, with $f_i > 0$, $i = 0, 1, ..., \sigma$. We define the differential operators $L_i(f_i, f_{i-1}, ..., f_0)$ by

$$L_0(f_0)(y) = f_0(x)\, y, \qquad L_i(f_i, ..., f_0)(y) = f_i(x) \frac{d}{dx} L_{i-1}(f_{i-1}, ..., f_0))(y),$$

$i = 1, 2, ..., \sigma$. We consider the pair of equations

$$L_n(r_n, r_{n-1}, ..., r_0)\, y + py = 0 \tag{1}$$

and

$$L_\nu(\rho_\nu, \rho_{\nu-1}, ..., \rho_0)\, y + qy = 0, \tag{2}$$

where $r_i, \rho_j > 0$, $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant \nu$, and where (without loss of generality) we assume $\int^\infty r_i^{-1}\, dt = \int^\infty \rho_j^{-1}\, dt = +\infty$, $i = 1, ..., n-1$, $j = 1, ..., \nu - 1$ [11], and $p, q \in C(I)$. Along with Eq. (1) we shall consider boundary conditions of the type

$$\begin{aligned} L_i\, y(a) &= 0, \qquad i \in \mathscr{I} \equiv \{i_1, ..., i_k\}, \\ L_j\, y(s) &= 0, \qquad j \in \mathscr{J} \equiv \{j_1, ..., j_{n-k}\} \end{aligned} \tag{3}$$

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with

$$0 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n-1, \qquad k \leqslant n-1, a, s \in I, \tag{4}$$
$$0 \leqslant j_1 < j_2 < \cdots < j_{n-k} \leqslant n-1.$$

Similar boundary conditions will be considered for Eq. (2) (with, $i_l, j_{v-l} \leqslant v-1$ in (4)). As in [1, 3] for $a \in I$ we define the ith extremal point $\theta_i(\mathscr{I}, \mathscr{J}; a)$ corresponding to Eq. (1) and the conditions (3) to be the ith value of $s \in I \cap (a, +\infty)$ for which there exists a nontrivial solution of (1), (3). This is, to be precise, the ith right extremal point. Similarly one may define the ith (left) extremal point. In [3] Elias investigated the connection between the oscillation of (1) and the extremal points under the assumption $(-1)^{n-k} p(x) < 0$. (This is a necessary condition for existence of extremal points if $p(x)$ is of one sign.) Two systems of boundary conditions of type (3) are especially important, namely focal type and conjugate type:

$$\begin{aligned} L_i y(a) &= 0, \qquad i = 0, 1, ..., k-1 \\ L_j y(s) &= 0, \qquad j = 0, ..., n-k-1 \end{aligned} \qquad \textit{conjugate type} \tag{5}$$

and

$$\begin{aligned} L_i y(a) &= 0, \qquad i = 0, 1, ..., k-1 \\ L_j y(s) &= 0, \qquad j = k, ..., n-k-1. \end{aligned} \qquad \textit{focal type} \tag{6}$$

The *first* extremal point for (5) is called a $(k, n-k)$ *conjugate point* and for (6) a $(k, n-k)$ *focal point.* These boundary conditions will be denoted by $(\mathscr{I}_c, \mathscr{J}_c)$, $(\mathscr{I}_f, \mathscr{J}_f)$, respectively. We say that Eq. (1) is $(\mathscr{I}, \mathscr{J})$-*disconjugate* on I in case the first extremal point $\theta_1(\mathscr{I}, \mathscr{J}; a)$ does not exist in $I \cap (a, +\infty)$ for any $a \in I$. We indicate this by writing $\theta_1(\mathscr{I}, \mathscr{J}; a) = \infty$.

In this paper we investigate the question: If Eq. (1) is $(\mathscr{I}, \mathscr{J})$-disconjugate then under what conditions will Eq. (2) be $(\hat{\mathscr{I}}, \hat{\mathscr{J}})$-disconjugate and how are n, v, $(\mathscr{I}, \mathscr{J})$ and $(\hat{\mathscr{I}}, \hat{\mathscr{J}})$ related? In such generality, the question is perhaps too difficult. In [7] Kim investigated comparison theorems for disfocality types for pairs of equations ((1, (2)) and in [1] the authors established integral comparison theorems for general boundary conditions under the assumption $n = v$ and $\rho_i = r_i$, $0 \leqslant i \leqslant n$. Special cases of these results were also considered in [3, 4, 5, 8, 9, 10]. We would like to combine the techniques of [1, 7]. However, it is necessary to restrict the class of boundary conditions of type (3). We say that the boundary conditions $(\mathscr{I}, \mathscr{J})$ (for (1)) are *admissible* in case for any integer l, $1 \leqslant l \leqslant n-1$, at least l terms of the sequence $i_1, ..., i_k, j_1, ..., j_{n-k}$ are less than l. We denote by $\mathscr{A}$ the class of all admissible boundary conditions. It has been shown (see [1] for further references) that $(\mathscr{I}, \mathscr{J}) \in \mathscr{A}$ if and only if for any $a, s \in I$, $a < s$ there exists no nontrivial solution of $L_n y = 0$ satisfying (3).

As in [1] we introduce the partial order $\prec$ on the set $\mathscr{A}$. To do this, we define the relation $\to$ on the class of all sets of indices $\{l_1, l_2, ..., l_q\}$, where $0 \leqslant l_1 < l_2 < \cdots < l_q \leqslant n-1$, $1 \leqslant q \leqslant n-1$, as follows: If $\mathscr{I}_1 = \{l_1, ..., l_q\}$, $\mathscr{I}_2 = \{m_1, ..., m_r\}$ then $\mathscr{I}_1 \to \mathscr{I}_2$ means that the two index sets agree except for one index $\hat{s}$ for which $m_{\hat{s}} = l_{\hat{s}} + 1 < l_{\hat{s}+1}$ (where if $\hat{s} = q$ we define $l_{\hat{s}+1} = \infty$). The multi-valued map $\sigma: \mathscr{A} \to 2^{\mathscr{A}}$ is defined by

$$\sigma(\mathscr{I}, \mathscr{J}) = \{(\hat{\mathscr{I}}, \hat{\mathscr{J}}) \in \mathscr{A} : \mathscr{I} \to \hat{\mathscr{I}} \text{ and } \mathscr{J} = \hat{\mathscr{J}} \text{ or } \mathscr{I} = \hat{\mathscr{I}} \text{ and } \mathscr{J} \to \hat{\mathscr{J}}\}.$$

If $(\mathscr{I}, \mathscr{J}), (\hat{\mathscr{I}}, \hat{\mathscr{J}}) \in \mathscr{A}$ and either $(\mathscr{I}, \mathscr{J}) \equiv (\hat{\mathscr{I}}, \hat{\mathscr{J}})$ or there exists an integer $m \geqslant 1$ and a sequence $\{(\mathscr{I}_r, \mathscr{J}_r)\}_{r=1}^m \subset \mathscr{A}$ with $(\mathscr{I}_1, \mathscr{J}_1) = (\mathscr{I}, \mathscr{J})$, $(\mathscr{I}_m, \mathscr{J}_m) = (\hat{\mathscr{I}}, \hat{\mathscr{J}})$, and $(\mathscr{I}_r, \mathscr{J}_r) \in \sigma(\mathscr{I}_{r-1}\, \mathscr{J}_{r-1})$, $r = 2, ..., m$, then we write $(\mathscr{I}, \mathscr{J}) \prec (\hat{\mathscr{I}}, \hat{\mathscr{J}})$. It is easy to check that $\prec$ defines a partial order on $\mathscr{A}$.

In Section 2 we collect some additional preliminary results from [1, 7] which will be needed to establish the main results in Section 3.

2. Preliminary Results

We recall a number of results from [1, 7] which will be needed subsequently. Unless specifically mentioned otherwise, we shall always assume in this paper that $(-1)^{n-k} p(x) < 0$ (for Eq. (1)) and $(-1)^{\nu - l} q(x) < 0$ (for Eq. (2)). It follows from [1, Lemma 2.1] that if $(\mathscr{I}, \mathscr{J}) \in \mathscr{A}$ then $(\mathscr{I}_c, \mathscr{J}_c) \prec (\mathscr{I}, \mathscr{J})$ and if $(\mathscr{I}, \mathscr{J}) \in \mathscr{A}$ with $\mathscr{I} = \{0, 1, ..., k-1\}$ $(= \mathscr{I}_f)$, then $(\mathscr{I}, \mathscr{J}) \prec (\mathscr{I}_f, \mathscr{J}_f)$. Moreover, from [1, Theorem 2.3], if $(\mathscr{I}, \mathscr{J}), (\hat{\mathscr{I}}, \hat{\mathscr{J}}) \in \mathscr{A}$ with $(\mathscr{I}, \mathscr{J}) \prec (\hat{\mathscr{I}}, \hat{\mathscr{J}})$ and if $\theta_1(\mathscr{I}, \mathscr{J}; a)$ exists for Eq. (1), then $\theta_1(\hat{\mathscr{I}}, \hat{\mathscr{J}}; a)$ exists and $\theta_1(\hat{\mathscr{I}}, \hat{\mathscr{J}}; a) \leqslant \theta_1(\mathscr{I}, \mathscr{J}; a)$. It follows therefore that the first extremal point $\theta_1(\mathscr{I}, \mathscr{J}; a)$ is a non-increasing function with respect to the partial order on $\mathscr{A}$. The next two results apply to the case when $\nu = n$ and $\rho_i = r_i$ in Eqs. (1) and (2). We denote by $\bar{\theta}_1(\mathscr{I}, \mathscr{J}; a)$ the first extremal point for Eq. (2).

Theorem 2.1 [1]. *Assume $\theta_1(\mathscr{I}, \mathscr{J}; a) < \infty$ for all $(\mathscr{I}, \mathscr{J}) \in \mathscr{A}$ and suppose*

$$\int_z^\infty \frac{|p(t)|}{r_0(t)\, r_n(t)}\, dt \leqslant \int_x^\infty \frac{|q(t)|}{r_0(t)\, r_n(t)}\, dt. \tag{7}$$

Then $\bar{\theta}_1(\mathscr{I}, \mathscr{J}; a) < \infty$ for all $(\mathscr{I}, \mathscr{J}) \in \mathscr{A}$.

THEOREM 2.2 [1]. *Let* $(\mathscr{I}, \mathscr{J}) \in \mathscr{A}$, $\mathscr{I} = \{0, 1, ..., k-1\}$, *and assume* (2) *is* $(\mathscr{I}, \mathscr{J})$ *disconjugate on* (a, ∞) *and that* (7) *holds. Then* (1) *is* $(\mathscr{I}, \mathscr{J})$ *disconjugate on* (a, ∞).

The proofs of Therorems 2.1 and 2.2 depend strongly on the fact (see [3]) that $\theta_1(\mathscr{I}_c, \mathscr{J}_c; a) = +\infty \Rightarrow \theta_1(\mathscr{I}_f, \mathscr{J}_f; a) = +\infty$. Before stating the final results of this section, we define the functions

$$\varphi(f_{i_0})(t) = \frac{1}{f_{i_0}(t)},$$

$$\varphi(f_{i_m}, ..., f_{i_0})(t) = \frac{1}{f_{i_m}(t)} \int_a^t \varphi(f_{i_{m-1}}, ..., f_{i_0})(s)\, ds, \qquad m \geqslant 1.$$

THEOREM 2.3 [7]. *Assume Eq.* (1) *is* $(k, n-k)$ *disfocal on* (a, ∞) *and assume* $n \geqslant m$, $k > m$. *Then the* m*th order equation*

$$L_m(r_n, ..., r_{n-m})y + \frac{\varphi(r_0, ..., r_k)}{\varphi(r_{n-m}, ..., r_k)}\, py = 0 \tag{8}$$

is $(k-n+m, n-k)$ *disfocal on* $[a, \infty)$.

THEOREM 2.4 [7]. *Assume* $n \geqslant m$ *and let the functions* $\zeta_0, ..., \zeta_{n-m-1}$ *be positive and continuous on* $[a, \infty)$. *If the* mth *order equation*

$$L_m(r_m, ..., r_0)\, y + \frac{\varphi(\zeta_0, ..., \zeta_{n-m-1}, r_0, ..., r_{j-1})}{\varphi(r_0, ..., r_{j-1})}\, py = 0 \tag{9}$$

is $(j, m-j)$ *disfocal on* (a, ∞) *then the* nth *order equation*

$$L_n(r_m, ..., r_0, \zeta_{n-m-1}, ..., \zeta_0)\, y + py = 0 \tag{10}$$

is $(n-m+j, m-j)$ *disfocal on* $[a, \infty)$.

THEOREM 2.5 [7]. *Assume* $n = \nu$, $\rho_i \geqslant r_i$, $i = 1, ..., n-1$, *and*

$$0 < \hat{q} \equiv \frac{(-1)^{n-k-1} q}{\rho_0 \rho_n} \leqslant \hat{p} \equiv \frac{(-1)^{n-k-1} p}{r_0 r_n} \tag{11}$$

and that Eq. (1) *is* $(k, n-k)$ *disfocal on* (a, ∞). *Then Eq.* (2) *is also* $(k, n-k)$ *disfocal on* (a, ∞).

3. Generalized Comparison Theorems

Our main focus in this section is the extension of the previous results in which pointwise inequalities are replaced by integral inequalities and by considering more general types of boundary conditions than those of [7].

Theorem 3.1. *Assume Eq.* (1) *is* $(\mathscr{I}_{\mathrm{f}}, \mathscr{J}_{\mathrm{f}})$*-disconjugate on* (a, ∞), *let* $n \geqslant m$, $k > n-m$, *and assume*

$$0 < \int_x^\infty \frac{|q(t)|\, dt}{r_{n-m}(t)\, r_n(t)} \leqslant \int_x^\infty \frac{\varphi(r_0, ..., r_k)}{r_{n-m}(t)\, r_n(t)\, \varphi(r_{n-m}, ..., r_k)} \,|p(t)|\, dt, \qquad x \in (a, \infty). \tag{12}$$

Then

$$L_m(r_n, ..., r_{n-m})\, y + qy = 0 \tag{13}$$

is $(k-n+m, n-k)$ *disfocal on* (a, ∞).

Proof. Since Eq. (1) is $(\mathscr{I}_{\mathrm{f}}, \mathscr{J}_{\mathrm{f}})$-disconjugate on (a, ∞) (i.e., (1) is $(k, n-k)$ disfocal on (a, ∞)) it follows by Theorem 2.3 that

$$L_m(r_n, ..., r_{n-m})\, y + \frac{\varphi(r_0, ..., r_k)}{\varphi(r_{n-m}, ..., r_k)}\, py = 0 \tag{14}$$

is $(k-n+m, n-k)$-disfocal on (a, ∞). Therefore, by Theorem 2.2 it follows from (12) that Eq. (13) is also $(k-n+m, n-k)$ disfocal on (a, ∞). This completes the proof.

As an interesting special case, we have

Corollary 3.2. *Assume that*

$$y^{(n)} + py = 0 \tag{15}$$

is $(k, n-k)$*-disfocal on* (a, ∞), *let* $n \geqslant m$, $k > n-m$, *and assume*

$$0 < \int_x^\infty |q(t)|\, dt \leqslant \int_x^\infty \frac{(k-n+m)!}{k!}\, t^{n-m}\, |p(t)|\, dt, \qquad x \in (a, \infty). \tag{16a}$$

Then

$$y^{(m)} + qy = 0 \tag{16b}$$

is $(k-n+m, n-k)$ *disfocal on* (a, ∞).

Proof. One need only observe that if $r_i = 1$, $i = 0, ..., n$ then $\varphi(r_0, ..., r_k(t)/\varphi(r_{n-m}, ..., r_k)(t) = (k-n+m)!\, t^{n-m}/k!$. The result then follows directly from Theorem 3.1.

As a further extension of Theorem 3.1 we have the following

THEOREM 3.3. *Assume Eq.* (1) *is* $(\mathscr{I}, \mathscr{J})$ *disconjugate on* (a, ∞) *where* $(\mathscr{I}, \mathscr{J}) \in \mathscr{A}$ *with* $\mathscr{I} = \mathscr{I}_f = \{0, 1, ..., k-1\}$. *Let* $n \geqslant m$, $k > n - m$ *and assume that* (12) *holds on* (a, ∞). *Then Eq.* (13) *is* $(\hat{\mathscr{I}}, \mathscr{J})$*-disconjugate on* (a, ∞) *where* $\hat{\mathscr{I}} = \{0, 1, ..., k-n+m-1\}$.

Proof. Since $\theta_1(\mathscr{I}, \mathscr{J}; a) = +\infty$, it follows from [1, Lemma 2.1] (and the remarks at the beginning of Section 2) that $\theta_1(\mathscr{I}_f, \mathscr{J}_f; a) = +\infty$. Hence, by Theorem 3.1 we have that Eq. (13) is $(\hat{\mathscr{I}}, \mathscr{J}_f)$-disconjugate on (a, ∞). Since $(\hat{\mathscr{I}}, \mathscr{J}) \prec (\hat{\mathscr{I}}, \mathscr{J}_f)$, it follows by [1, Theorem 2.3] that Eq. (13) is $(\hat{\mathscr{I}}, \mathscr{J})$-disconjugate on (a, ∞). This completes the proof.

In the next three results, we assume that $n = v$.

THEOREM 3.4. *Assume* $n = v$, $\rho_i(t) \geqslant r_i(t)$, $i = 1, ..., n-1$, $\rho_0(t)\, \rho_n(t) \geqslant r_0(t)\, r_n(t)$, $t \in (a, \infty)$ *and assume further that Eq.* (1) *is* $(\mathscr{I}_f, \mathscr{J}_f)$*-disconjugate on* (a, ∞). *Then Eq.* (2) *is also* $(\mathscr{I}_f, \mathscr{J}_f)$*-disconjugate on* (a, ∞) *provided*

$$\int_x^\infty \frac{|q(t)|\, dt}{\rho_0(t)\, \rho_n(t)} \leqslant \int_x^\infty \frac{|p(t)|\, dt}{\rho_0(t)\, \rho_n(t)}, \qquad x \in (a, \infty). \tag{17}$$

Proof. Since (1) is $(\mathscr{I}_f, \mathscr{J}_f)$-disconjugate on (a, ∞) it follows from Theorem 2.5 (with $q = p$) that

$$L_n(\rho_n, ..., \rho_0)\, y + py = 0 \tag{18}$$

is also $(\mathscr{I}_f, \mathscr{J}_f)$-disconjugate on (a, ∞). Now from inequality (17) and Theorem 2.1 we have that Eq. (2) is $(\mathscr{I}_f, \mathscr{J}_f)$-disconjugate on (a, ∞). This completes the proof.

COROLLARY 3.5. *Assume all conditions of Theorem* 3.4 *except that Eq.* (1) *is assumed to be* $(\mathscr{I}, \mathscr{J})$ *disconjugate on* (a, ∞), *where* $\mathscr{I} = \mathscr{I}_f$, $(\mathscr{I}, \mathscr{J}) \in \mathscr{A}$. *Then Eq.* (2) *is also* $(\mathscr{I}_f, \mathscr{J})$ *disconjugate on* (a, ∞).

Proof. Since $\theta_1(\mathscr{I}, \mathscr{J}; a) = +\infty$ and $(\mathscr{I}_c, \mathscr{J}_c) \prec (\mathscr{I}, \mathscr{J})$ it follows from the remarks in Section 2 that $\theta_1(\mathscr{I}_c, \mathscr{J}_c; a) = +\infty$. Therefore, $\theta_1(\mathscr{I}_f, \mathscr{J}_f; a) = +\infty$ from [3] and so by Theorem 3.4 it follows that $\bar{\theta}_1(\mathscr{I}_f, \mathscr{J}_f; a) = +\infty$ where $\bar{\theta}_1(\mathscr{I}, \mathscr{J}; a)$ denotes the first extremal point corresponding to Eq. (2). But then from $(\mathscr{I}, \mathscr{J}) \prec (\mathscr{I}_f, \mathscr{J}_f)$, we have $\bar{\theta}_1(\mathscr{I}, \mathscr{J}; a) = +\infty$. That is, Eq. (2) is $(\mathscr{I}_f, \mathscr{J})$ disconjugate on (a, ∞).

THEOREM 3.6. *Assume* $n = \nu$, $\rho_i(t) \leqslant r_i(t)$, $i = 1, .., n-1$, *and* $\rho_0(t)\,\rho_n(t) \leqslant r_0(t)\,r_n(t)$, $t \in (a, \infty)$. *Assume further that*

$$\int_x^\infty \frac{|q(t)|\,dt}{r_0(t)\,r_n(t)} \geqslant \int_x^\infty \frac{|p(t)|\,dt}{r_0(t)\,r_n(t)}, \qquad x \in (a, \infty) \tag{19}$$

and that $\theta_1(\mathscr{I}, \mathscr{J}; a) < +\infty$ *for all* $(\mathscr{I}, \mathscr{J}) \in \mathscr{A}$. *Then* $\bar{\theta}_1(\mathscr{I}, \mathscr{J}; a) < +\infty$ *for all* $(\mathscr{I}, \mathscr{J}) \in \mathscr{A}$.

Proof. If there exists $(\hat{\mathscr{I}}, \hat{\mathscr{J}}) \in \mathscr{A}$ with $\bar{\theta}_1(\hat{\mathscr{I}}, \hat{\mathscr{J}}; a) = +\infty$, then since $(\mathscr{I}_c, \mathscr{J}_c) \prec (\hat{\mathscr{I}}, \hat{\mathscr{J}})$, it follows that $\bar{\theta}_1(\mathscr{I}_c, \mathscr{J}_c; a) = +\infty$ and hence $\bar{\theta}_1(\mathscr{I}_f, \mathscr{J}_f; a) = +\infty$ (by [3]). But then by Theorem 3.4 we must have $\theta_1(\mathscr{I}_f, \mathscr{J}_f; a) = +\infty$, which is a contradiction.

In the next two results we compare Eqs. (1) and (2) with n and ν not necessarily equal. For the set $\mathscr{I} = \{i_1, ..., i_q\}$ we will use the notation $\|\mathscr{I}\| = q$ and similarly, if $\mathscr{J} = \{j_1, ..., j_s\}$, $\|\mathscr{J}\| = s$.

THEOREM 3.7. *Assume that Eq.* (1) *is* $(\mathscr{I}_f, \mathscr{J})$ *disconjugate on* (a, ∞) *with* $\|\mathscr{I}_f\| = k$, $\|\mathscr{J}\| = n-k$. *Assume further that there exists an integer* m *satisfying* $n-k < m \leqslant \min\{n, \nu\}$ *and such that*

(i) $\rho_{m-i}(t) \geqslant r_{n-i}(t)$, $i = 1, ..., m-1$

(ii) $$\frac{\varphi(\rho_\nu, \rho_{\nu-1}, ..., \rho_{m-n+k+1})}{\rho_m \rho_0 \varphi(\rho_m, \rho_{m-1}, ..., \rho_{m-n+k+1})} \leqslant \frac{\varphi(r_0, ..., r_k)}{r_n r_{n-m} \varphi(r_{n-m}, r_{n-m+1}, ..., r_k)}$$

(iii) $$\int_x^\infty \frac{|q(t)|\,dt}{\rho_\nu(t)\,\rho_0(t)} \leqslant \int_x^\infty \frac{|p(t)|\,dt}{\rho_\nu(t)\,\rho_0(t)}.$$

Then

$$L_\nu(\rho_\nu, ..., \rho_0)\,y + (-1)^{m+\nu} q y = 0 \tag{20}$$

is $(\hat{\mathscr{I}}_f, \hat{\mathscr{J}})$ *disconjugate on* (a, ∞) *where*

$$\|\hat{\mathscr{I}}_f\| = k - n + m \qquad \text{and} \qquad \|\hat{\mathscr{J}}\| = n - k + \nu - m.$$

Proof. Since Eq. (1) is $(\mathscr{I}_f, \mathscr{J})$ disconjugate and $(\mathscr{I}_f, \mathscr{J}) \prec (\mathscr{I}_f, \mathscr{J}_f)$ it follows that $\theta_1(\mathscr{I}_f, \mathscr{J}_f; a) = +\infty$. Now by [7, Theorem 4a] conditions (i) and (ii) imply that

$$L_\nu(\rho_\nu, ..., \rho_0)\,y + (-1)^{m+\nu} p y = 0 \tag{21}$$

is $(k-n+m,\ n-k+\nu-m)$-disfocal on (a, ∞). Then by Theorem 3.4, we have because of condition (iii), that Eq. (20) is also $(k-n+m,$ $n-k+\nu-m)$ disfocal on (a, ∞). Therefore if $\|\hat{\mathscr{I}}_f\| = k-n+m$, $\|\hat{\mathscr{J}}\| =$

$n-k+v-m$, then $(\hat{\mathscr{I}}_f, \hat{\mathscr{I}}) \prec (\hat{\mathscr{I}}_f, \hat{\mathscr{I}}_f)$ so that Eq. (20) is also $(\hat{\mathscr{I}}_f, \hat{\mathscr{I}})$ disconjugate on (a, ∞). This completes the proof.

In a similar manner, by applying Theorems 2.4 and 3.4, one may obtain the following result. We omit the proof.

THEOREM 3.8. *Assume* $n \geqslant m$ *and let the functions* $\zeta_0, ..., \zeta_{n-m-1}$ *be positive and continuous on* $[a, \infty)$. *Assume that*

$$L_m(r_m, ..., r_0)\, y + \frac{\varphi(\zeta_0, ..., \zeta_{n-m-1}, r_0, ..., r_{j-1})}{\varphi(r_0, ..., r_{j-1})}\, py = 0 \tag{22}$$

is $(\mathscr{I}_f, \mathscr{I})$*-disconjugate with* $\|\mathscr{I}_f\| = j$, $\|\mathscr{I}\| = m-j$. *Assume further that*

$$\int_x^\infty \frac{|q(t)|\, dt}{r_m(t)\, \zeta_0(t)} \leqslant \int_x^\infty \frac{|p(t)|\, dt}{r_m(t)\, \zeta_0(t)}, \qquad x \in (a, \infty).$$

Then the nth *order equation*

$$L_n(r_m, ..., r_0, \zeta_{n-m}, ..., \zeta_0)\; y + qy = 0 \tag{23}$$

is $(\hat{\mathscr{I}}_f, \hat{\mathscr{I}})$*-disconjugate on* (a, ∞) *where* $\|\hat{\mathscr{I}}_f\| = n-m+j$, $\|\hat{\mathscr{I}}\| = m-j$.

In our final result we would like to relax somewhat the strict sign assumptions by applying results of [2].

THEOREM 3.9. *Assume that* $n = v$ *and that Eq.* (1) *is* $(\mathscr{I}_f, \mathscr{I}_f)$*-disconjugate on* (a, ∞). *Assume also that* $(-1)^{n-k} q(x) < 0$, $x \in (a, \infty)$, $\rho_i \geqslant r_i$, $i = 1, ..., n-1$, $\rho_0 \rho_n \geqslant r_0 r_n$, *and that*

$$0 < \int_x^\infty \frac{|q(t)|\, dt}{r_0(t)\, r_n(t)} \leqslant \int_x^\infty \frac{p(t)\, dt}{r_0(t)\, r_n(t)}. \tag{24}$$

Then Eq. (2) *is* $(\mathscr{I}_f, \mathscr{I}_f)$ *disconjugate on* (a, ∞).

Proof. From [2, Theorem 2.4], we conclude that

$$L_n(r_n, ..., r_0)\, y + qy = 0 \tag{25}$$

is $(\mathscr{I}_f, \mathscr{I}_f)$-disconjugate on (a, ∞) because of (24). Therefore from Theorem 2.5 (with $p = q$ in (11)), it follows that Eq. (2) is $(\mathscr{I}_f, \mathscr{I}_f)$-disconjugate on (a, ∞). This completes the proof.

Remark. Additional extensions of the results of [1, 2, 7] are also possible in much the same manner. In [6], various comparison results are also obtained by combining the integral comparison results of [1] with the nth order Euler equation.

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